bras of index $m = 3$. Here we have to consider only the linear equation $P_1 = 0$ in (6). Without any further adjunction, we obtain an element $A \neq 0$ in D, with a characteristic equation $x^3 - h_3 = 0$. This gives Wedderburn's result that all normal division algebras of index 3 are cyclic.

In the case $m = 4$, we again consider only the equation $P_1 = 0$ instead of (6). We obtain an element $A \neq 0$ in D, which has a characteristic equation of the form $x^4 - h_2x^2 - h_4 = 0$. The construction of such an element A forms the main part of Albert's proof of the theorem that every normal division algebra of index $m = 4$ possesses a Galois splitting field of degree 4 and can, therefore, be written as a crossed product, provided the characteristic of F is different from 2.

[1] For references to the theory of algebras, cf. M. Deuring's book: *Algebren* (*Ergebnisse der Mathematik*, Berlin, 1935).

[2] J. L. M. Wedderburn, *Trans. Amer. Math. Soc.*, **22**, 129 (1931).

[3] A. A. Albert, *Trans. Amer. Math. Soc.*, **31**, 253 (1929) and *Bull. Amer. Soc.*, **38**, 703 (1932).

[4] R. Brauer, *Math. Zeitschr.*, **30**, 79 (1929).

[5] This can also be seen from general considerations concerning simple algebras, without using the explicit form (1) of the elements.

[6] A. A. Albert, *Trans. Amer. Math. Soc.*, **36**, 885 (1934). For the case of a field F of characteristic 5, cf. A. A. Albert, *Trans. Amer. Soc.*, **39**, 183 (1936).

THE LIFT AND DRAG FUNCTIONS FOR AN ELASTIC FLUID IN TWO DIMENSIONAL IRROTATIONAL FLOW

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1. The lift function Y and the drag function X are defined by the equation

$$dX = (p + \rho u^2)dy - \rho uvdx = pdy + ud\psi = (p + \rho q^2)dy - \rho vd\phi,$$
$$dY = \rho uvdy - (p + \rho v^2)dx = vd\psi - pdx = \rho ud\phi - (p + \rho q^2)dx, \quad (1)$$

where u, v are the component velocities, p is the pressure and ρ is the density at the point x, y. The symbol q^2 is written for $u^2 + v^2$. Putting $x = r\cos\theta$, $y = r\sin\theta$, $v_r = u\cos\theta + v\sin\theta$, $v_\theta = v\cos\theta - u\sin\theta$ we have

$$dX = (p_\theta \sin\theta - \rho v_r v_\theta \cos\theta)dr + (p_r \cos\theta - \rho v_r v_\theta \sin\theta)rd\theta,$$
$$dY = (p_r \sin\theta + \rho v_r v_\theta \cos\theta)rd\theta - (p_\theta \cos\theta + \rho v_r v_\theta \sin\theta)dr, \quad (2)$$

where $p_r = p + \rho v_r^2$, $p_\theta = p + \rho v_\theta^2$.

Generalizing the analysis of Glauert and Lamb we write

$$\begin{aligned}\phi &= V[r\cos\theta + A_0 + A_1\log r + A_2(\log r)^2 + \ldots \\ &\qquad + (1/r)\{B_0 + B_1\log r + B_2(\log r)^2 + \ldots\} + \ldots] \\ \psi &= \rho_\infty V[r\sin\theta + C_0 + C_1\log r + C_2(\log r)^2 + \ldots \qquad (3) \\ &\qquad + (1/r)\{D_0 + D_1\log r + D_2(\log r)^2 + \ldots\} + \ldots],\end{aligned}$$

where the coefficients A_s, B_s, C_s, D_s are all functions of θ.

Differentiating these expressions with respect to r and θ we find

$$\begin{aligned}v_r &= \frac{\partial\phi}{\partial r} = V[\cos\theta + (1/r)\{A_1 + 2A_2\log r + 3A_3(\log r)^2 + \ldots\} \\ &\qquad + (1/r)^2\{(B_1 - B_0) + (2B_2 - B_1)\log r + \ldots\} + \ldots], \\ v_\theta &= (1/r)\frac{\partial\phi}{\partial\theta} = V[-\sin\theta + (1/r)\{A_0' + A_1'\log r + A_2'(\log r)^2 + \ldots\} \\ &\qquad + (1/r)^2\{B_0' + B_1'\log r + B_2'(\log r)^2 + \ldots\} + \ldots], \\ \rho v_r &= (1/r)\frac{\partial\psi}{\partial\theta} = \rho_\infty V[\cos\theta + (1/r)\{C_0' + C_1'\log r + C_2'(\log r)^2 + \ldots\} \\ &\qquad + (1/r)^2\{D_0' + D_1'\log r + D_2'(\log r)^2 + \ldots\} + \ldots] \qquad (4) \\ \rho v_\theta &= -\frac{\partial\psi}{\partial r} = \rho_\infty V[-\sin\theta - (1/r)\{C_1 + 2C_2\log r + 3C_3(\log r)^2 + \ldots\} \\ &\qquad + (1/r)^2\{D_0 - D_1 + (D_1 - 2D_2)\log r + (D_2 - 3D_3)(\log r)^2 + \ldots\} + \ldots],\end{aligned}$$

where primes denote differentiations with respect to θ.

We shall endeavor to express as many coefficients as possible in terms of A and C. To find the expressions we shall assume that

$$\begin{aligned}\rho &= \rho_\infty[1 + R_0(1/r) + R_1(1/r)\log r + R_2(1/r)(\log r)^2 + \ldots \\ &\qquad + S_0(1/r)^2 + S_1(1/r)^2\log r + S_2(1/r)^2(\log r) + \ldots] \qquad (5)\end{aligned}$$

where the coefficients R_n, S_n are all functions of θ.

Using this expression for ρ to obtain series for ρv_r and ρv_θ from those for v_r and v_θ and then comparing the chief terms in the two sets of expansions we obtain a set of equations of which only four will be written down here as sufficient for our present needs but it may be mentioned that the other equations may be needed for a later calculation of the moment on a stationary obstacle.

$$\begin{aligned}R_0\cos\theta + A_1 &= C_0', &\qquad R_1\cos\theta + 2A_2 &= C_1', \qquad (6) \\ -R_0\sin\theta + A_0' &= -C_1, &\qquad -R_1\sin\theta + A_1' &= -2C_2.\end{aligned}$$

To obtain another set of equations we shall suppose that the pressure and density are connected by the relation $p = f(\rho) - \rho f'(\rho)$ so that if c is the local velocity of sound, $c^2 = dp/d\rho = -\rho f''(\rho)$. We shall suppose, moreover, that the arbitrary constant in $f'(\rho)$ is chosen so that the equation of Bernoulli is $^1/_2 q^2 = f'(\rho)$. Then by Taylor's expansion, if $a^2 = (dp/d\rho)_\infty = -\rho_\infty f''(\rho_\infty)$, we have

$$^1/_2 q^2 = {}^1/_2 V^2 + (\rho - \rho_\infty) f''(\rho_\infty) + \ldots$$

Substituting the expression for $\rho - \rho_\infty$ and equating coefficients of $(1/r)$, $(1/r)\log r$, etc., we find that

$$V^2(A_1 \cos\theta - A_0' \sin\theta) = -a^2 R, \; V^2(2A_2 \cos\theta - A_1' \sin\theta) = -a^2 R_1 \quad (7)$$

the other equations being omitted though they may be needed in a calculation of the moment.

Writing $A_1 = A(a^2 - V^2 \sin^2\theta)$, $C_1 = C(a^2 - V^2\sin^2\theta)$ we find that

$$A_0' = -V^2 A \sin\theta\cos\theta - a^2 C, \; C_0' = (a^2 - V^2)A - V^2 C \sin\theta\cos\theta. \quad (8)$$

If the total circulation is $-K$, a finite quantity, we find that

$$K = -\int_0^{2\pi} v_\theta r d\theta = -V\int_0^{2\pi} A_0' d = V^3 \int_0^{2\pi} A \sin\theta\cos\theta \, d\theta + a^2 V \int_0^{2\pi} C d\theta \quad (9)$$

and that A_1, A_2, ... must be uniform functions of θ.

Furthermore, if there is no flow of matter to infinity on the whole C_0, C_1, C_2, ... must be uniform functions of θ and so, in particular,

$$0 = \int_0^{2\pi} C_0' d\theta \;\therefore\; (a^2 - V^2)\int_0^{2\pi} A d\theta = V^2 \int_0^{2\pi} C \sin\theta\cos\theta d\theta. \quad (10)$$

We also find from equations (6) and (7) that

$$\begin{aligned} R_0 &= -V^2(A\cos\theta + C\sin\theta), \\ R_1(a^2 - V^2\cos^2\theta) &= V^2 A_1' \sin\theta - V^2 C_1' \cos\theta, \\ 2A_2(a^2 - V^2\cos^2\theta) &= a^2 C_1' - V^2 A_1' \sin\theta\cos\theta, \\ -2C_2(a^2 - V^2\cos^2\theta) &= A_1'(a^2 - V^2) + C_1' V^2 \sin\theta\cos\theta, \\ p &= p_\infty + a^2(\rho - \rho_\infty) + \ldots \\ &= p_\infty - \rho_\infty V^2(A_1\cos\theta - A_0'\sin\theta)(1/r) \\ &\quad -\rho_\infty V^2(2A_2\cos\theta - A_1'\sin\theta)(1/r)\log r \text{ approximately.} \end{aligned} \quad (11)$$

Combining the last equation with equations obtained by forming the squares and products of the series in equations (4) we find that to a first approximation

$$p_r = p_\infty + \rho_\infty V^2[\cos^2\theta + (1/r)(C_0'\cos\theta + A_0'\sin\theta) + (1/r)(\log r)(C_1'\cos\theta + A_1'\sin\theta)]$$

$$\rho v_r v_\theta = \rho_\infty V^2[-\sin\theta\cos\theta + (1/r)(A_0'\cos\theta - C_0'\sin\theta) + (1/r)(\log r)(A_1'\cos\theta - C_1'\sin\theta) \qquad (12)$$

$$p_\theta = p_\infty + \rho_\infty V^2[\sin^2\theta + (1/r)(C_1\sin\theta - A_1\cos\theta) + (1/r)\log r\,(2C_2\sin\theta - 2A_2\cos\theta)].$$

Hence to a first approximation

$$dX = dr[(p_\infty + \rho_\infty V^2)\sin\theta + \rho_\infty V^2 C_1(1/r)] + d\theta\,[p_\infty r\cos\theta + \rho_\infty V^2(r\cos\theta + C_0' + C_1'\log r)]$$

$$X = (p_\infty + \rho_\infty V^2)r\sin\theta + \rho_\infty V^2(C_0 + C_1\log r); \qquad (13)$$

thus X is a uniform function of θ.

$$\begin{aligned} -dY = dr[&p_\infty\cos\theta + (1/r)\\ &(c_1\sin\theta\cos\theta - A_1\cos^2\theta + A_0'\sin\theta\cos\theta - C_0'\sin^2\theta) + (1/r)\\ &\log r\,(2C_2\sin\theta\cos\theta - 2A_2\cos^2\theta + A_1'\sin\theta\cos\theta - C_1'\sin^2\theta)\\ &+ d\theta[p_\infty r\sin\theta + \rho_\infty V^2(A_0' + A_1'\log r + \ldots)] \qquad (14)\\ &Y = p_\infty r\cos\theta - \rho_\infty V^2\{A_0 + A_1\log r + \ldots\}. \end{aligned}$$

Since A_1 is a uniform function of θ the total lift is $KV\rho_\infty$ as in the theorems of Lord Rayleigh, Kutta, Joukowsky and Glauert.

2. We now introduce the functions $\chi = ux + vy - \phi$, $\Omega = \rho uy - \rho vx - \psi$, which occur in the theory of the Legendre transformation of the partial differential equations satisfied by ϕ and ψ. Their approximate expressions are

$$\chi = r\frac{\partial\phi}{\partial r} - \phi = V[A_1 - A_0 + (2A_2 - A_1)\log r + \ldots + (1/r)\{B_1 - 2B_0 + (2B_2 - 2B_1)\log r + \ldots\} + \ldots] \qquad (15)$$

$$\Omega = r\frac{\partial\psi}{\partial r} - \psi = \rho_\infty V[C_1 - C_0 + (2C_2 - C_1)\log r + \ldots + (1/r)\{D_1 - 2D_0 + (2D_2 - 2D_1)\log r + \ldots\} + \ldots].$$

We see that both χ and Ω are generally infinite for large values of r. When r is kept constant Ω is a uniform function of θ but χ is not; the change in χ as θ increases from 0 to 2π is for large values of r equal to K.

It should be remarked that if we write $u = q\cos\omega$, $v = q\sin\omega$, the functions χ and Ω are connected by the equations

$$\frac{\partial\Omega}{\partial\omega} = -q\rho\frac{\partial\chi}{\partial q}, \quad \frac{\partial\chi}{\partial\omega} = q\frac{\partial\Omega}{\partial(\rho q)}. \qquad (16)$$

3. If, by means of equations (1) we express dX and dY in terms of $d\phi$ and $d\psi$ we obtain the equations

$$\begin{aligned} pd\phi &= vdX - udY, \\ (p + \rho q^2)d\psi &= \rho(udX + vdY), \\ \rho q^2 dX &= p\rho v d\phi + (p + \rho q^2)ud\psi, \\ \rho q^2 dY &= -p\rho u d\phi + (p + \rho q^2)vd\psi \end{aligned} \tag{17}$$

which may be compared with the equations

$$\begin{aligned} d\phi &= udx + vdy, \\ d\psi &= \rho udy - \rho vdx, \\ \rho q^2 dx &= \rho ud\phi - vd\psi, \\ \rho q^2 dy &= \rho vd\phi + ud\psi. \end{aligned} \tag{18}$$

Introducing a fictitious elastic fluid in the XY-plane and using large letters for the quantities relating to this fluid except for the density which we denote by σ, we have relations

$$\begin{gathered} P = -1/p, \quad P + \sigma Q^2 = -1(p + \rho q^2), \quad Q = q/p, \\ C^2 = \frac{dP}{d\sigma}, \qquad 1 - \frac{Q^2}{C^2} = \left(1 - \frac{q^2}{c^2}\right)\left(\frac{p}{p + \rho q^2}\right)^2 \\ \frac{dP}{dQ} = -\frac{q\rho}{p + \rho q^2} = -Q\sigma. \end{gathered} \tag{19}$$

The last equation, which corresponds to $dp/dq = -q\rho$ shows that P, Q, σ are connected by the same relation as the pressure, velocity and density in an elastic fluid. We also have the relations

$$\begin{aligned} p(p + \rho q^2)dx &= \rho uvdX - (p + \rho u^2)dY, \\ p(p + \rho q^2)dy &= (p + \rho v^2)dX - \rho uvdY, \end{aligned} \tag{20}$$

$$\begin{gathered} P + \sigma U^2 = -\frac{p + \rho u^2}{p(p + \rho q^2)}, \quad P + \sigma V^2 = -\frac{p + \rho v^2}{p(p + \rho q^2)}, \\ \sigma UV = -\frac{\rho uv}{p(p + \rho q^2)} \end{gathered}$$

which show that there is a complete reciprocity between the real fluid and the fictitious fluid. Another correspondence between two elastic fluids may be obtained by introducing two arbitrary constants h, H and writing $x' = x + hY$, $y' = y - hX$, $X' = X + Hy$, $Y' = Y - Hx$. We then have the relations

$$vdX' - udY' = (p + H)d\phi, \quad \rho udX' + \rho vdY' = (p + \rho q^2 + H)d\psi$$

$$vdy' + udx' = (1 - hp)d\phi, \quad \rho udy - \rho vdx' = 1 - h(p + \rho q^2)d\psi. \quad (21)$$

Writing $\phi' = \phi$, $\psi' = (1 + hH)\psi$ and introducing primed quantities for a fictitious fluid in which ψ' is the stream function and ϕ' the velocity potential, we have the relations

$$p' = \frac{p + H}{1 - ph}, \quad u' = \frac{u}{1 - ph}, \quad v' = \frac{v}{1 - ph}, \quad q' = \frac{q}{1 - ph}$$

$$\rho' = \rho \frac{(1 - ph)(1 + Hh)}{1 - h(p + \rho q^2)}, \qquad p' + \rho' q'^2 = \frac{p + \rho q^2 + H}{1 - h(p + \rho q^2)}$$

$$\frac{dp'}{dp} = \frac{1 + hH}{(1 - ph)^2}, \qquad \frac{dp'}{dq'} = -\rho' q'$$

$$1 - \frac{q'^2}{c'^2} = \left(1 - \frac{q^2}{c^2}\right) \frac{(1 - hp)^2}{[1 - h(p + \rho q^2)]^2}\,. \quad (22)$$

The last relation shows that $q'^2 > c'^2$ when $q^2 > c^2$. This is to be expected from the general behavior of characteristics in a point transformation.

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